Introduction to Differential Geometry

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Abstract

In these short notes, we introduce the basics of Riemannian geometry. After defining Riemannian manifolds, which are smooth manifolds equipped with a Riemannian metric, we go on to define the Levi-Civita connection, which is a canonical connection associated to a Riemannian manifold. The Levi-Civita connection enables us to define parallel transport of a vector along a curve, as well as the notion of a geodesic. A geodesic on a Riemannian manifold can be defined as a curve that locally minimizes the energy of a curve. The Riemann curvature ten- sor of a Riemannian manifold is the obstruction for a general parallel transported vector along a small closed loop to return to itself. It thus measures intuitively how far a Riemannian manifold is from be- ing "flat", where the prototypical example of a flat space is Euclidean space. We finally write down Einstein's equations in vacuum, i.e. with $T_{ab} = 0$, with a cosmological constant.

1 Affine Connections, Parallel Transport and Geodesics

An undergraduate course in differential geometry following say Do Carmo's "Differential Geometry of Curves and Surfaces", would study curves and surfaces embedded in Euclidean 3-space $\mathbb{E}^3$. The latter has a natural inner product, namely

$$< u, v >= u_1 v_1 + u_2 v_2 + u_3 v_3.$$

For a surface S embedded in $\mathbb{E}^3$, the inner product on $\mathbb{E}^3$ restricts to a "field" of inner products at the tangent space of each point $p \in S$, namely

$$g_p(u_p, v_p) =< u_p, v_p >,$$

where u_p, $v_p \in T_p(S)$. The notion of a Riemannian metric g on a smooth manifold M generalizes such a field.

Definition 1.1. A *Riemannian metric* g on a smooth manifold M is a "collection" (or "field") of inner products g_p on each tangent space $T_p(M)$ that depends smoothly on p. The latter condition can be made precise in the following way: you require that $g(X, Y)$ be a smooth function on M whenever X and Y are smooth tangent vector fields on M.

Theorem 1.2. *A smooth manifold always admits a Riemannian metric (our manifolds are assumed to be paracompact, by definition).*

The proof of this existence statement follows from the existence of a so-called "partition of unity". Intuitively, you choose an open cover of M by charts such that each $p \in M$ has a neighborhood that intersects only finitely many open sets belonging to that open cover. You then choose a Riemannian metric on each element of the open cover, and then glue them together using a partition of unity to get a smooth globally defined Riemannian metric on M. On any smooth manifold M , if $f : U \to \mathbb{R}$ is a smooth function defined in some neighborhood U of $p \in M$, then one can define the directional derivative of f at p in some direction $v \in T_p(M)$. You just take any smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M$$

such that $\gamma(0) = p$ and $\gamma'(0) = v$, and then define

$$D_u(f)(p) = \frac{d}{dt}(f(\gamma(t)))_{t=0}.$$

One can then check that this definition is independent of the choice of γ. However, on a smooth manifold M, if one attempts a similar approach to differentiate a given smooth vector field Y, at some point p in a given direction $v \in T_p(M)$, one immediately runs into a problem: for two neighboring points p and q, there is no unique and canonical way to identify $T_p(M)$ and $T_q(M)$.

Manifolds, maps, vector fields and so on will be assumed to be smooth, i.e. C^∞, unless stated otherwise, and so the word "smooth" will be omitted.

We have the following definition.

Definition 1.3. An affine connection ∇ on a smooth manifold M is a smooth map from $\mathfrak{X} \times \mathfrak{X}$ into $\mathfrak{X}$, where the latter denotes the space of (smooth) vector fields on M, such that:

$$\begin{aligned} (X, Y) &\mapsto \nabla_X(Y) \\ \nabla_X(Y_1 + Y_2) &= \nabla_X(Y_1) + \nabla_X(Y_2) \\ \nabla_X(fY) &= \nabla_X(f)Y + f\nabla_X(Y) \\ \nabla_{X_1+X_2}(Y) &= \nabla_{X_1}(Y) + \nabla_{X_2}(Y) \\ \nabla_{fX}(Y) &= f\nabla_X(Y), \end{aligned}$$

for all vector fields X, X_1, X_2, Y, Y_1, Y_2 and all functions $f : M \to \mathbb{R}$.

The space of affine connections on a smooth manifold M is an affine space modelled on the space of 1-forms on M with values in $\mathrm{End}(TM)$, and is therefore infinite-dimensional. However, if M is equipped with a Riemannian metric g, in which case we say that the pair (M, g) is a Riemannian manifold, then one can pinpoint a unique affine connection D called the Levi-Civita connection by imposing two additional conditions. But first, we need a couple of definitions.

Definition 1.4. The *torsion* tensor τ^∇ of an affine connection ∇ on a smooth manifold M is defined by

$$\tau^\nabla(X,Y) = \nabla_X(Y) = \nabla_Y(X) - [X,Y],$$

where $[X,Y]$ is the Lie bracket of X and Y. An affine connection ∇ is said to be torsion-free if its torsion τ^∇ vanishes identically.

Definition 1.5. An affine connection ∇ on a Riemannian manifold (M,g) is said to be *compatible* with g if the following holds:

$$\nabla_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all vector fields X, Y and Z.

We then have the following result.

Theorem 1.6. *There is a unique affine connection on a Riemannian manifold (M,g) which is both torsion-free and compatible with g. This affine connection is known as the Levi-Civita connection and will be denoted by D.*

Proof. First, we assume that there is such an affine connection D, and prove that it is unique. Consider

$$\begin{aligned} D_Z(g(X,Y)) &= g(D_Z X, Y) + g(X, D_Z Y) \\ D_X(g(Y,Z)) &= g(D_X Y, Z) + g(Y, D_X Z) \\ D_Y(g(Z,X)) &= g(D_Y Z, X) + g(Z, D_Y X). \end{aligned}$$

If we add the first two equations and then subtract the third, and use the torsion-free condition, we obtain:

$$\begin{aligned} D_Z(g(X,Y)) + D_X(g(Y,Z)) - D_Y(g(Z,X)) =& 2g(D_Z X, Y) + \dots \\ &+ g(Y,[X,Z]) + g(X,[Z,Y]) + g(Z,[X,Y]) \end{aligned}$$

Solving for $g(D_Z X, Y)$ yields

$$\begin{aligned} 2g(D_Z X, Y) =& D_Z(g(X,Y)) + D_X(g(Y,Z)) - D_Y(g(Z,X)) - \dots \\ &- g(Y,[X,Z]) - g(X,[Z,Y]) - g(Z,[X,Y]) \end{aligned}$$

This shows uniqueness, since the right-hand side does not involve the covariant derivative of a vector field. It remains to show existence. For that purpose, one can define a connection by the previous formula, and check that it is well defined, and is torsion-free and compatible with g. □

Definition 1.7. If (M,∇) is a manifold with connection, then given a smooth curve $\gamma : [0,a] \to M$ and an initial tangent vector $v \in T_{\gamma(0)}M$, one can define the *parallel transport* $v(t)$ of v along the curve γ by requiring that $v(0) = v$ and that $v(t)$ be covariantly constant along γ or, in other words, that

$$\nabla_{\frac{d}{dt}}(v)(t) = 0.$$

Definition 1.8. If (M, ∇) is a manifold with connection, then a curve $\gamma : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, is said to be a *geodesic* if its velocity vector is parallel, i.e. that

$$\nabla_{\frac{d}{dt}} \gamma'(t) = 0$$

for all $t \in I$.

This may not be, as it is, a very enlightening enlightening definition but, in the case of a Riemannian manifold (M, g), with its Levi-Civita connection D, there is a nice characterization of geodesics as critical points of the energy functional on the space of curves between two fixed points p and q in M. Let

$$\Omega(p,q) = \{\gamma : [0,1] \to M \text{ piecewise smooth } | \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

We define the energy functional $E : \Omega(p,q) \to \mathbb{R}$ by

$$E(\gamma) = \int_0^1 ||\gamma'(t)||^2 dt.$$

The Euler-Lagrange equation associated to the functional E is precisely the geodesic equation.

2 The Riemann Curvature Tensor

From now on, we will work on a Riemannian manifold (M, g). Suppose we were to parallel transport a vector v_0 along a small loop γ, and call the end vector v_1. In general, v_1 may be different from v_0. Convince yourself by considering a closed piecewise smooth loop on a sphere or in the hyperbolic plane. There is an obstruction for parallel transport along small closed piecewise smooth loops to be the identity: it is called the Riemann curvature tensor. This is perhaps the way to define the Riemann curvature tensor that gives the most insight into what it is. We will define it the following way though, which is easier to use.

Definition 2.1. The Riemann curvature tensor R is defined as

$$R(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

That R is a tensor of type $(3,1)$ is easy to checked. The Riemann curvature tensor satisfies the following symmetries:

$$g(R(Y,X)Z,W) = g(R(X,Y)Z,W)$$
$$g(R(X,Y)W,Z) = -g(R(X,Y)Z,W)$$
$$g(R(Z,W)X,Y) = g(R(X,Y)Z,W).$$

Moreover, it satisfies the following identity, called the first Bianchi identity:

$$g(R(X,Y)Z,W) + g(R(Y,Z)X,W) + g(R(Z,X)Y,W) = 0.$$

These are the algebraic symmetries of the Riemann curvature tensor. It also satisfies a first order differential equation, called the second Bianchi identity:

$$(D_Z R)(X, Y) + (D_X R)(Y, Z) + (D_Y R)(Z, X) = 0.$$

Next, we define the Ricci tensor as the following trace of the Riemann curvature tensor:

Definition 2.2. The *Ricci tensor* $\mathrm{Ric}(-, -)$ of a Riemannian manifold (M, g) is defined by

$$\mathrm{Ric}(Y, Z) = \mathrm{Trace}(X \mapsto R(X, Y)Z).$$

It follows from the symmetries of the Riemann curvature tensor that the Ricci tensor is symmetric in its two arguments

$$\mathrm{Ric}(Y, X) = \mathrm{Ric}(X, Y),$$

for all vector fields X and Y. Finally, we define the scalar curvature s.

Definition 2.3. The *scalar curvature* s of a Riemannian manifold (M, g) is defined by

$$s(p) = \sum_{i=1}^{n} \mathrm{Ric}_p(e_i, e_i),$$

where $e_1, \ldots, e_n$ is an orthonormal frame of tangent vectors at p.

One can easily check that this definition is independent of the choice of orthonormal frame, and that the scalar curvature s is a smooth function of $p \in M$. We can now define Einstein manifolds.

Definition 2.4. A Riemannian manifold (M, g) is said to be *Einstein* if

$$\mathrm{Ric} = \Lambda g,$$

for some constant Λ.

Einstein manifolds are of interest to mathematicians and physicists alike. Their behavior depends strongly on whether Λ is positive, zero or negative.